Macroscopic Causality and Permanence of Smoothness for Two-Particle Scattering*

David N. Williams

The Institute for Advance Study
Princeton, New Jersey†

August, 1967‡

Abstract

Recent developments in the formulation of causality restrictions on the S matrix are reviewed, with attention focused on the behavior of matrix elements of the translation operator between suitably localized in and out states. Rapid decrease for large translations outside the timelike velocity cone of the center of momentum follows from Poincaré invariance and boundedness of S, as a result of a generalization of a theorem of Jost and Hepp. At present, rapid decrease can be proved in the Haag-Ruelle scattering theory, when the in state is translated to large positive times, but not for the remaining timelike directions, where thresholds of intermediate particles play a role. In the case of two-particle reactions, we show that rapid decrease for timelike directions is equivalent to permanence of smoothness of the p-space wave function, as an application of rapid convergence properties of the angular momentum expansion.

*Work supported by the National Science Foundation.

†Present address: Randall Laboratory of Physics, The University of Michigan, Ann Arbor, Michigan 48109 –1040.

‡July 3, 2011: This is a LaTeX version of extended notes for lectures presented by the author at the Fourth Boulder Conference on Particles and High Energy Physics, 1967. It has cosmetic changes from the published version in *Lectures in Theoretical Physics, XB*, (Gordon and Breach, New York, 1968), pp. 357–376.

Contents

1 Introduction

The consensus for some time, mainly due to experience with dispersion relations, has been that the analytic stucture of scattering amplitudes in some way expresses their causal content. More recently some quantitative meaning has been given to this notion, growing out of the Coleman-Norton [1, 2] interpretation of "nonnegative-α" type Landau singularities as corresponding to kinematically and causally realizable, multiple scattering processes. The concept of macroscopic causality for the S matrix of strongly interacting particles has been given an impressively complete formulation in studies of F. Pham [3] and of C. Chandler and H. P. Stapp [4], as a catalog of rapid cluster properties, corresponding to the kinematical and causal possibility or impossibility of having contributions to a given scattering process from multiple scattering events, when the space-time scale is made large enough. The vacuum cluster property [5, 6] (connectedness structure) and the one-particle pole structure [7, 8, 9, 10] of the S matrix are special cases that have been studied extensively.

Pham argues that if the p-space, connected scattering amplitudes are boundary values of analytic functions, if the physical region boundary values are analytic except for nonnegative-α type Landau singularities, and if the physical region discontinuities across these singularities (with the $+i\epsilon$ prescription) are given by the Cutkosky rules, then the amplitudes, smeared with suitable smooth wave functions, have the rapid, multiscattering decomposition properties that are identified with macrocausality [3].

Chandler and Stapp, on the other hand, take macrocausality as a postulate, and prove that the connected amplitudes are C^∞ in the physical region, except on the Landau surfaces. If they assume in addition that the amplitudes are analytic in the physical region whenever they are C^∞, they also get the $+i\epsilon$ prescription for continuation around the Landau singularities [4]. On their line of reasoning, the $+i\epsilon$ prescription can be expected to follow in the sense of distribution theory even without the assumption of analyticity at points of infinite differentiability. The physical region singularities (points where the amplitude is not C^∞) should be parametrized by connected, mass-shell multiple scattering amplitudes, but with all internal mass-shell delta functions replaced by Feynman propagators with the usual $+i\epsilon$.

We find the formulation of macrocausality proposed by these authors appealing. If we accept it, the relation between physical region differentiability and macrocausal properties of scattering amplitudes can be systematically understood. But we also feel that not all rapid cluster laws in the list are equally compellng to the intuition, a feeling that roughly parallels Chandler and Stapp's classification of the laws of macrocausality according to *weak asymptotic causality* (WAC) and *strong asymptotic causality* (SAC) [4]. In Section 2, we illustrate the contention that there is an overlap between those macrocausal laws about whose common sense status we might hesitate (SAC) and those laws that have so far resisted proof in axiomatic field theory. Macrocausality is naturally associated with the idea of a *coarse-grained* space-time structure [7, 9, 11], which we might at first glance think abstracts the physically nonobjectionable content from the field-theoretic idea of microscopic space-time. Actually, it is unclear how much macrocausality is guaranteed, if the S matrix can be interpolated by local or almost local fields. The rapid cluster properties derived in the Hagg-Ruelle scattering theory [12, 13] for a class of spacelike separations, including those that increase the impact parameter, by K. Hepp [6], are convincing evidence that an "almost local" Wightman field theory of massive particles is indeed a theory of short-range interactions (if there is any interaction at all); but strong timelike cluster properties are generally much more difficult to prove, because they involve the multiparticle structure of the S matrix, corresponding to physical region Landau singularities, in an essential way. So far, Hepp has obtained results in the cases of the vacuum and one-particle structures, for "causally independent" configurations [8].

In this paper, we study the subset of laws of macrocausality associated with rapid decrease of matrix elements of the operator for timelike and spacelike translations between in and out states that are smooth and have compact support in p-space; and in the case of two-particle scattering, we relate them to a regularity condition which says roughly that the S matrix preserves the smoothness of the in wave function, except at normal thresholds. For simplicity, we neglect spin. To define our notation, let $\mathcal{F}$ be the Fock space for one type of massive, spinless particle:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \,, \tag{1.1}$$

where $\mathcal{H}_n$ is the Hilbert space of free, symmetric wave functions $g(\mathbf{p}_1, \ldots, \mathbf{p}_n)$, with the usual Lorentz invariant scalar product. The S matrix is a unitary map of $\mathcal{F}$ onto itself, which commutes with the Fock representation $U(b, \Lambda)$ of the identity component of the Poincaré group, $\mathcal{P}_+$. We write for the space-time translations

$$T(b) \equiv U(b, I) = \exp(iP\cdot b)\,, \tag{1.2}$$

where for $g \in \mathcal{H}_n$,

$$\begin{aligned} T(b) &= \exp\left(i\,\Sigma p_j \cdot b\right)\, g \\ p_j^0 &= \omega_j \equiv +\sqrt{m^2 + \mathbf{p}_j^2}\,. \end{aligned} \tag{1.3}$$

We look at matrix elements of the form

$$\langle f \,|\, S\,T(b)\,|g\rangle = \langle f \,\mathsf{out}|\, T(b)\,|g\,\mathsf{in}\rangle\,, \tag{1.4}$$

where $f, g \in \mathcal{D}(\mathbb{R}^{3\ell}), \mathcal{D}(\mathbb{R}^{3n})$, the Schwartz spaces [14] of C^∞ functions of compact support. It would not modify the discussion much to let f and g be in the Schwartz spaces $\mathcal{S}(\mathbb{R}^{3\ell})$ and $\mathcal{S}(\mathbb{R}^{3n})$ of C^∞ functions which decrease rapidly with all derivatives near infinity, but it's convenient to use functions of type $\mathcal{D}$.

In Section 2, we classify the laws of rapid decrease according to their interpretation on the basis of macrocausality. We point out that when b increases along space-time directions that lie outside the timelike *velocity cone* where the center of momentum of the free system propagates, rapid decrease of the matrix elements follows already from the $\mathcal{P}_+$ invariance and unitarity (or just boundedness) of S, along with the exclusion of massless particles from the theory, by a generalization of a theorem of Jost and Hepp [15], on matrix elements of the translation operator between states generated from the vacuum by polynomials in the field operators, to matrix elements between C^∞ vectors of a unitary, continuous representation of $\mathcal{P}_+$.[1] For timelike directions inside the center of momentum velocity cone which, roughly speaking, delay the interaction of the incoming particles until after the outgoing particles have already emerged, thus violating elementary notions of causality, rapid decrease can be proved in the Haag-Ruelle scatting theory, if the particles have disjoint velocities, as a straightforward application of methods developed by K. Hepp. We do not give the proof; no new ideas are involved. This is a special, but characteristic case of WAC, in the terminology of Chandler and Stapp, which is thus supported by field theory. For the remaining timelike directions, we give a physical motivation for the fact that the laws of decrease depend inextricably on regularity properties of the S matrix which have not yet been proved in local or quasilocal field theory. This is a special case of SAC.

Our main technical result is a theorem that pins down the relation between rapid decrease of the matrix elements for timelike translations and regularity of the S matrix in the simplest, nontrivial case of two-particle scattering. To write down the theorem, we need some more notation.

If we let E_2 be the projection operator for two-particle states in $\mathcal{F}$, the two-particle S matrix is $S_2 = E_2 S E_2$. It maps $\mathcal{H}_2$ into $\mathcal{H}_2$. Although we do not assume that S_2 has normal threshold singularities in the variable $s = (p_1 + p_2)^2$, we want to be able to exclude them in case it does, so we consider a fixed, compact set Δ of values of the total four-momentum P; and we let $\mathcal{D}_\Delta$ be the set of C^∞ $g \in \mathcal{H}_2$ with support Δ in the variables P, $\operatorname{supp}_P g = \Delta$. Then $\mathcal{D}_\Delta \subset \mathcal{D}(\mathbb{R}^6)$. Finally, it turns out that we do not have to consider *all* timelike translations; it is enough to look at the one-parameter family of *proper time* translations with respect to the center of momentum, $\exp(iM\tau)$, where $M = +\sqrt{P\cdot P}$ is self-adjoint, and $-\infty < \tau < \infty$.

In Section 4, we prove the "only if" part of the following theorem. The "if" part is easy, so we prove it further on in this introduction.

Theorem 1. Let Δ exclude the two-particle threshold; i.e., if $P \in \Delta$, then $s = P^2 \neq 4m^2$. Then

$$\left\langle f \,\middle|\, S_2\, e^{iM\tau} \,\middle|\, g \right\rangle \in \mathcal{S}(\mathbb{R}) \quad \text{for all} \quad f, g \in \mathcal{D}_\Delta\,, \tag{1.5}$$

[1] Although it does not, as far as we know, appear in the literature, some form of this generalization was already known, and was applied for example, by Hepp [8]. The version here evolved from a collaboration (unpublished) between J. Bros and the author.

if and only if $S_2\,\mathcal{D}_\Delta \subset \mathcal{D}_\Delta$ and $S_2^\dagger\,\mathcal{D}_\Delta \subset \mathcal{D}_\Delta$. That is, rapid decrease of $F(\tau) \equiv \left\langle f \middle| S_2\, e^{iM\tau} \middle| g \right\rangle$ is equivalent to the permanence of smoothness of two-particle states under the action of S_2 and $S_2^\dagger$.[2]

To prove this theorem, we use rapid convergence properties of the angular momentum expansion for states in $\mathcal{D}_\Delta$, above the two-particle threshold, which we give without proof in Section 3. We also derive there, as an immediate consequence of the rapid decrease of $F(\tau)$, that the partial wave amplitudes are C^∞ in s. As A. Martin has recently emphasized [16], it has not been proved in field theory that the partial waves, defined as boundary values from above in the cut s plane, are even continuous anywhere in the elastic region. Thus, the law of macrocausality corresponding to rapid decrease of $F(\tau)$ is conceivably stronger at this point than microscopic causality and the other ingredients of local field theory.

The permanence of smoothness condition, although not enough to give differentiability of the invariant amplitude $a(s,t)$, certainly implies more than the smoothness of the partial waves. It can be shown that it retricts the growth in total angular momentum of arbitrary derivatives of the partial waves to that of a polynomial in J.

It is easy to see that permanence of smoothness is equivalent to a six-parameter, spatial cluster property, where we translate each of the in and out particles by its own spatial translation. For example, let

$$g^{\mathbf{a}_1,\mathbf{a}_2} \equiv \exp -i\left(\mathbf{p}_1\cdot\mathbf{a}_1 + \mathbf{p}_2\cdot\mathbf{a}_2\right)\, g \tag{1.6}$$

and let $S_2^\dagger f \equiv w \in \mathcal{D}_\Delta$. Then

$$G(\mathbf{a}_1,\mathbf{a}_2) \equiv \langle f |\, S\, | g^{\mathbf{a}_1,\mathbf{a}_2}\rangle \tag{1.7}$$

is the Fourier transform of the smooth function $\overline{w}g$, up to smooth energy factors; and hence it vanishes rapidly for large $\mathbf{a}_1$ or $\mathbf{a}_2$. Conversely, if $G(\mathbf{a}_1,\mathbf{a}_2)$ vanishes rapidly at infinity for all f and g in $\mathcal{D}_\Delta$, it follows that $\overline{w}g$, and hence $\overline{w}$, is smooth.

Of course the rapid decrease of $G(\mathbf{a}_1,\mathbf{a}_2)$ has a macrocausal interpretation; and since the theorem tells us that it is equivalent to rapid decrease of $F(\tau)$, the interpretations of the two laws ought to be related. The relation is discussed in Section 2.

The following corollary makes the connection with matrix elements of $T(b)$:

Corollary 1. $\langle f | S \exp iM\tau | g\rangle \in \mathcal{S}(\mathbb{R})$, for all f and $g \in \mathcal{D}_\Delta$, if and only if $\langle f | S\, T(b) | g\rangle \in \mathcal{S}(\mathbb{R}^4)$, for all f and $g \in \mathcal{D}_\Delta$.

We prove the corollary, and incidentally the "if" part of the theorem, in two steps:

(i) Permanence of smoothness implies that $\mathcal{T}(b) \equiv \langle f | S\, T(b) | g\rangle \in \mathcal{S}(\mathbb{R}^4)$. That is because

$$\mathcal{T}(b) = \int \frac{d^3p_1}{2\omega_1}\,\frac{d^3p_2}{2\omega_2}\;\bar{f}\, h\, e^{iP\cdot b}\,, \tag{1.8}$$

[2] Differentiability in τ is automatic, because M can be applied arbitrarily many times to f or g, leaving them in $\mathcal{D}_\Delta$.

where $h \equiv S_2\, g$ is smooth. By the implicit function theorem, this can be written as the Fourier transform of a C^∞ function with compact support in the variables P, as long as Δ does not contain the two-particle threshold, since P is smooth in $\mathbf{p}_1$ and $\mathbf{p}_2$, and $\partial P_\mu / \partial \mathbf{p}_i$, $i = 1, 2$, has rank four.

(ii) If $\mathcal{T}(b) \in \mathcal{S}(\mathbb{R}^4)$, then $\bar{f}\, h$ is smooth in P, and hence in $\sqrt{s}, \mathbf{P}$, because the change of variables $\sqrt{s} \leftrightarrow P^0$ is smooth whenever $s > 0$, as it is in the physical range. Then $F(\tau) \in \mathcal{S}(\mathbb{R})$ because it is the Fourier transform of a smooth function in $\sqrt{s}$ with compact support. That proves the "if" part of the theorem and of the corollary. It also proves the "only if" part of the corollary, because if $F(\tau) \in \mathcal{S}(\mathbb{R})$, then the theorem tells us that we have permanence of smoothness; and thus $\mathcal{T}(b)$ is in $\mathcal{S}(\mathbb{R}^4)$.

It takes more work to prove that rapid decrease of $F(\tau)$ implies that S_2 and $S_2^\dagger$ preserve smoothness. That is what we do in Section 4.

A last introductory remark: all of our results go through for particles with different nonvanishing masses and spins, as long as we are careful about choosing the spin conventions with respect to which smoothness is defined. The canonical and spinor conventions are suitable, but the helicity convention is not [17].

2 Macrocausal Interpretation

The intuitive foundation for the formulation of macrocausal properties of the S matrix of massive particles is the notion that, because the interactions are short range, the regions and times of interaction of a collection of incoming and outgoing particles, and the orbits along which they propagate before or after collision, can be adequately described on a large space-time scale by the free, multiparticle wave functions, if they are sufficiently well localized.

Macrocausality is a list of statements about what happens to the transition amplitudes $\langle f\ \mathsf{out} | g\ \mathsf{in} \rangle$ when the in and out particles are grouped according to regions and times of interaction, which are then moved away from each other by large, four-dimensional translations [3, 4]. We distinguish two types of statement:

(i) If the regions and times of interaction are separated from each other in such a way that it is always possible to connect them with intermediate, real particles, satisfying the mass shell and four-momentum conservation constraints at each "vertex" (region of interaction), and moving forward in time on classical, free orbits, the transition amplitude should factorize for large separations (up to a phase depending on the separation) into a product of scattering amplitudes, one for each region of interaction, with integrations over the allowed internal momenta.

(ii) If the regions and times of interaction are separated in such a way that it is impossible for physical particles, propagating forward in time and conserving four-momentum, to communicate between them, the amplitude should go to zero for large separations.

The rate at which the amplitude factorizes or vanishes should depend, insofar as the principle of short range is valid, on how well the free wave functions localize the external particles, i.e., on how well they restrict the interactions to finite regions and times. The first type of statement is always strong, in the sense of SAC, while the second can be either strong or weak.

In this paper, we are directly concerned only with the particularly simple subset of laws of type (ii) where all in particles interact in one space-time region, and all out particles in another. Thus, we study the decrease of matrix elements for large, four-dimensional translations between suitably localized in and out states. Before we can discuss these laws more quantitatively, we must specify what "suitably localized" means: in what sense are the space-time regions of interaction defined when we choose our p-space wave functions to be of Schwartz class $\mathcal{D}$? We call the localization provided by these wave functions *Ruelle localization*, because it derives from his lemma on smooth solutions of the Klein-Gordon equation [13].

For one-particle states, it goes as follows: Let g be in $\mathcal{D}(\mathbb{R}^3)$, and let

$$\widetilde{g}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{\mathrm{d}^3 p}{2\omega}\, e^{-ip\cdot x}\, g(\mathbf{p})\,. \tag{2.1}$$

Ruelle's lemma says that the particle essentially propagates inside its four-dimensional velocity cone, which is defined by

$$\mathcal{C}(g) \equiv \left\{ x \,:\, \mathbf{x} = \frac{\mathbf{p}}{\omega}\, x_0\,, \text{with } \mathbf{p} \in \operatorname{supp} g \right\}. \tag{2.2}$$

For technical reasons, we follow a formulation of K. Hepp [18], and actually consider the velocity cone $\mathcal{C}_\eta(g)$ generated by the closure η of some open neighborhood of $\operatorname{supp} g$. The relevant part of Ruelle's lemma says that, for every integer N, there is a bound C_N, depending only on g and η, such that

$$|\widetilde{g}(x)| \leq C_N \left(1 + \mathbf{x}^2 + x_0^2\right)^{-2N} \tag{2.3}$$

for $x \notin \mathcal{C}_\eta(g)$. For $x \in \mathcal{C}_\eta(g)$, the decrease is like $\|x\|^{-3/2}$, where the norm is Euclidean.

When $x_0 = 0$, we can think of $\widetilde{g}$ as being localized in some finite region centered at the origin, the extent of which is restricted by the infinite set of bounds C_N. This kind of localization is somewhat crude, but it is sufficient for many arguments where one needs to know only that the region of localization is roughly finite, if arbitrarily large. To avoid a possible point of confusion, note that the translated wave function g^b is still in $\mathcal{D}$, and can be thought of either as localized when $x_0 = b_0$ in a region centered at $\mathbf{x} = \mathbf{b}$, whose size is restricted by the same C_N as before, or as localized when $x_0 = 0$ in a region restricted by new bounds $C_N(b)$.

One side remark about Ruelle localilzation: it makes no essential distinction between Newton-Wigner [19] space-time variables and ordinary configuration variables. The difference in p-space is a factor $\sqrt{\omega}$, which is smooth, and leaves $\mathcal{D}$ invariant. The effect is only to change the constants C_N. In our case, we can think of the configuration variables x unambiguously as translation variables, writing

$$\widetilde{g}(x) = \langle \varphi |\, T(x)\, | g \rangle\,, \tag{2.4}$$

where $\varphi(\mathbf{p}) = 1$ in supp g.

Ruelle localization of free states with several particles can be discussed in an analogous way. To the support of each $g \in \mathcal{D}(\mathbb{R}^{3n})$ we can associate a $3n$ (space-) + 1 (time-) dimensional velocity cone in the obvious way, outside of which $\widetilde{g}(t, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ vanishes rapidly. In the spirit of our discussion, the regions and times of interaction are essentially defined by the points in this velocity cone where two or more one-particle variables take the same value. If all the velocities $\mathbf{v}_i = \mathbf{p}_i/\omega_i$ are disjoint (*nonoverlapping* in the terminology of Hepp [18]) in supp g, i.e., $\mathbf{v}_i \neq \mathbf{v}_j$ if $i \neq j$, the particles are essentially together only at $t = 0$ and $\mathbf{x} = 0$; they are converging from $t = -\infty$ or diverging toward $t = +\infty$. Our picture then is that the particles can interact only within some roughly finite time interval about $t = 0$ within a roughly finite region centered at $\mathbf{x} = 0$. If two of the particles can have the same velocity, they can be together always; and we cannot say that the time interval of interaction is essentially bounded.

In the discussion below, it is convenient to assign a velocity cone to the total four-momentum of the system. There should be no confusion if we use the same notation for that as for the one-particle case, and write $C(g)$ for the timelike cone generated by $\mathbf{V} = \mathbf{P}/P_0$, with $P \in \mathrm{supp}_P\, g$ being the total four-momentum corresponding to momenta in supp g.

Now consider matrix elements of the form

$$\mathcal{T}(b) = \langle f\ \mathsf{out}|\, T(b)\, |g\ \mathsf{in}\rangle\,, \tag{2.5}$$

where $f \in \mathcal{D}(\mathbb{R}^{3\ell})$, $g \in \mathcal{D}(\mathbb{R}^{3n})$, $\ell, n \geq 2$, and the velocities are disjoint in supp f and supp g. We think of the out particles as interacting essentially at $t = 0$ and $\mathbf{x} = 0$, while the in particles interact essentially at $t = b_0$ and $\mathbf{x} = \mathbf{b}$. We choose the center of momentum velocity cones to be the same for f and g, i.e., $\mathrm{supp}_P\, f = \mathrm{supp}_P\, g$, with no loss of generality, because S conserves P.

Parametrizing the translations by $b = u\lambda$, where $\|u\|^2 \equiv u_0^2 + \mathbf{u}^2 = 1$, $\lambda \geq 0$, we look at three cases:

I. u is outside the center of momentum velocity cone $C_\eta(g)$ (now η is the closure of an open neighborhood of $\mathrm{supp}_P\, g$);

II. u is inside $C(g)$ and $u_0 > 0$;

III. u is inside $C(g)$, $u_0 < 0$, and $\mathrm{supp}_P\, g$ contains no normal thresholds.

In all three cases, as λ becomes large compared to the sizes of the regions and time intervals of interaction, which are "fixed" by f and g, the in and out regions can only be connected by free, intermediate particles if they travel with a velocity $\mathbf{u}/u_0$. They must all move along together; i.e., they must be at the normal threshold for the multiparticle intermediate state.

In case I that is essentially impossible, because u lies outside the velocity cone where the center of momentum, and hence any intermediate particle at a threshold, essentially propagates. It becomes impossible in case II because the in region gets retarded relative to the out region, so that any intermediate particles would have to propagate backward in time. More directly, we would see out particles emerging before the in particles had interacted. In the last case, intermediate particles could propagate forward in time, but there we have ruled out normal thresholds, preventing them from propagating together as they must.[3]

These arguments ought to be only as good as the localization of f and g. The formulation of macrocausality in cases I–III takes that into account. It says that $\mathcal{T}(u\lambda)$ decreases rapidly in λ:

$$|\mathcal{T}(u\lambda)| \leq C_N(1+\lambda)^{-N}. \tag{2.6}$$

To complete the mathematical statement, we ought to say how the rate of decrease depends on the direction u. Intuition need not be a reliable guide for such technical details; but to be able to prove anything, we should make sufficiently strong assumptions. We require that the decrease be uniform in u, i.e., that the bounds C_N be independent of u, when normal thresholds are excluded from $\operatorname{supp}_P g$; and if thresholds are allowed, so only cases I and II apply, we require uniform decrease outside the *backward* velocity cone generated by some $\eta \supset \operatorname{supp}_P g$. The latter condition can be justified in the Haag-Ruelle theory; both can be justified in perturbation theory.

Although we think the principles of macrocausality just reviewed qualify as highly reasonable properties of the S matrix, a word of caution is in order. Aside from introducing practically unavoidable technical elements, we swindled slightly in our intuitive discussion of case III. In fact, cases I–III are listed in what seems to be the order of decreasing intrinsic plausibility, i.e., the order of increasing restrictiveness. That agrees with Chandler and Stapp's classification, in which case II belongs to WAC and case III to SAC. Let's discuss it a bit more.

Rapid decrease in case I is *a posteriori* about as plausible as it can be; we can *prove* it just from the Poincaré invariance and boundedness of S.[4] This follows from a theorem of Jost and Hepp on matrix elements of the operator for spacelike translations in Wightman field theory [15]. By partially imitating their proof, one can free it from its field theory context to get the following.[5]

Let $U(b, A)$ be a continuous, unitary representation of the covering group of $\mathcal{P}_+$ on a Hilbert space $\mathcal{H}$, with infinitesimal generators P_μ and $M_{\mu\nu}$. Let $\mathcal{K}$ be the set of C^∞ vectors of the representation; i.e., all polynomials in the infinitesimal generators are defined on $\mathcal{K}$. It follows that $\mathcal{K}$ is dense in $\mathcal{H}$, because it contains all analytic vectors of U [20]. The terminology comes from the fact that if $\psi \in \mathcal{K}$ then $U(b, A)\psi$ is a C^∞, vector-valued function on the group manifold.[6] We assume that the spectrum of the total four-momentum P lies in the closed, forward light cone; and to each $\psi \in \mathcal{K}$ we assign its support in that spectrum, $\operatorname{supp}_P \psi$. We define the corresponding velocity cone $C_\eta(\psi)$ just as before, except for a technical restriction on the choice of η, which is automatically satisfied when $\operatorname{supp}_P \psi$ is compact, so that we do not need to worry about it for our application.

---

[3] This kind of argument owes a great deal to R. Norton's discussion of the physical interpretation of normal threshold singularities in [1].

[4] Even if S is only a partial isometry and not unitary, it is of course bounded.

[5] Although it does not, as far as we know, appear in the literature, some form of this generalization was already known, and was applied, for example, by Hepp in [8]. The version here evolved from a collaboration (unpublished) between J. Bros and the author.

[6] The Gårding domain provides at least a dense supply of typical C^∞ vectors. See [21].

Theorem 2. Let $\phi, \psi \in \mathcal{K}$. Then

$$|\langle\phi|\, T(u\lambda)\, |\psi\rangle\,| \le C_N(1+\lambda)^{-N} \tag{2.7}$$

uniformly for all directions which are outside $C_\eta(\psi)$ (for any fixed η) in the sense that $P_0\mathbf{u} \neq u_0\mathbf{P}$ for any $P \in \eta$.

Of course if $P = 0$ is in the support, no u satisfies the condition of the theorem. If $P = 0$ is not in the support, the support is bounded away from zero, since it's closed by definition. Because P has no spacelike vectors in its spectrum, we always have rapid decrease for spacelike directions.

In case I, macrocausality is a corollary to this theorem:

Corollary 2. Let S be bounded, let $U(b,\Lambda)$ be the Fock representation of $\mathcal{P}_+$ for spinless, massive particles, and let

$$[\, S,\, U(b,\Lambda)\,] = 0\,. \tag{2.8}$$

Let $f \in \mathcal{D}(\mathbb{R}^{3\ell})$, $g \in \mathcal{D}(\mathbb{R}^{3n})$. Then $\langle f\,|\, S\, T(u\lambda)\, |g\rangle$ vanishes rapidly in λ, uniformly for $u \notin C_\eta(g)$.

To prove the corollary, we need only note:

(a) f and g are C^∞ vectors, because if we use the explicit expressions for P_μ and $M_{\mu\nu}$ in the Fock representation we find that they leave Schwartz spaces of type $\mathcal{D}$ invariant;

(b) any bounded operator which commutes with $U(b,\Lambda)$ also commutes with P_μ and $M_{\mu\nu}$ (in the sense of Riesz and Sz.-Nagy [22]), and hence leaves $\mathcal{K}$ invariant.

Actually, we have proved a bit more than we demanded in case I. The corollary above makes no restriction that the velocities be disjoint.

In case II, we expect macrocausality to be more restrictive; and Chandler and Stapp [4] have indeed obtained interesting results from WAC in general. The technical statement appears not to be unreasonably restrictive because, as we mentioned in the Introduction, it can be proved in the Haag-Ruelle scattering theory. The essential ingredients in the method due to Hepp [8] are the spectrum condition (mass gap), and quasilocality of the fields in the sense of Haag-Ruelle. Besides, Chandler and Stapp have proved WAC in potential theory [4].

So far, axiomatic field theory has been unable to tell us anything about case III. We cannot expect the result to follow just from locality and simple properties of the momentum spectrum. It ought to depend on what intermediate states can occur, and on the absence of physical region singularities except on Landau surfaces, which is a relatively deep, nonlinear property of field theory, if it can be proved at all. The theorem described in the Introduction tells us precisely what regularity properties are needed for two-body scattering.

We can already understand that result, if we push our intuitive discussion a bit further in case III. Earlier we said that, as the in region of interaction gets advanced in time relative to the out region, it becomes impossible for the transition to go because not all of the intermediate particles can arrive at the out region within a sufficiently bounded time interval, if normal thresholds are excluded. For this argument to be valid, the intermediate particles must be sufficiently well localized on their classical orbits. In other words, their wave function (whatever *that* means) ought to be smooth in p-space (it automatically has compact support). Thus it is plausible that rapid decrease should hold only if the S matrix sufficiently preserves the smoothness of the in state.

To conclude our interpretive review, we mention the intuitive relation between the timelike cluster properties II and III and spacelike cluster laws where the in or out particles are individually translated. A mathematical equivalence was pointed out in the Introduction, for two-particle reactions, which we look at to get the idea. Define $G(\mathbf{a}_1, \mathbf{a}_2)$ as in Eq. (1.7). If the two-particle threshold is excluded from the support, the velocities are automatically disjoint. Parametrize the translations by

$$\mathbf{a}_i = \lambda \mathbf{e}_i\,, \quad i = 1, 2\,, \quad \lambda \geq 0\,; \qquad |\mathbf{e}_1|^2 + |\mathbf{e}_2|^2 = 1\,. \tag{2.9}$$

A special case of a result of K. Hepp [6] shows that, in the Haag-Ruelle theory, $G(\lambda\mathbf{e}_1, \lambda\mathbf{e}_2)$ decreases rapidly in λ for all directions $(\mathbf{e}_1, \mathbf{e}_2)$ where $\mathbf{e}_1 - \mathbf{e}_2$ does not point along a relative velocity $\mathbf{v}_1 - \mathbf{v}_2$ in $\operatorname{supp} g$. Thus, if $\mathbf{e}_1 - \mathbf{e}_2$ points along some $\mathbf{v}_1 - \mathbf{v}_2$ in $\operatorname{supp} g$, and if $\operatorname{supp} g$ is small enough not to contain $(-\mathbf{p}_1, -\mathbf{p}_2)$, we get rapid decrrease for $G(-\lambda\mathbf{e}_1, -\lambda\mathbf{e}_2)$, but not for $G(+\lambda\mathbf{e}_1, +\lambda\mathbf{e}_2)$.

These two situations correspond, respectively, to cases II and III. For simplicity, set $\mathbf{e}_2 = 0 = \mathbf{v}_2$; and consider $G(-\lambda\mathbf{e}_1, 0)$. According to Fig. 1, the region where the in particles interact gets effectively delayed, as in case II. Either for timelike translations, as in II, or for the relative spatial translation here, rapid decrease is true in field theory.

For the other direction, of course, the interaction of the in particles gets effectively advanced, as in Fig. 2, corresponding to case III. Rapid decrease has been proved in field theory neither for $G(+\lambda\mathbf{e}_1, +\lambda\mathbf{e}_2)$ nor for case III. The obstacles to proof are presumably the same, and we should not hope for rapid decrease unless all thresholds are excluded from the support.

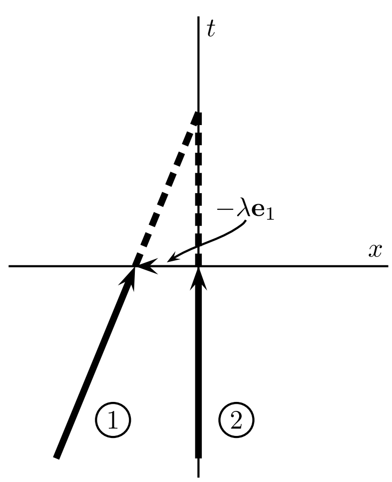

Figure 1. Case II: WAC

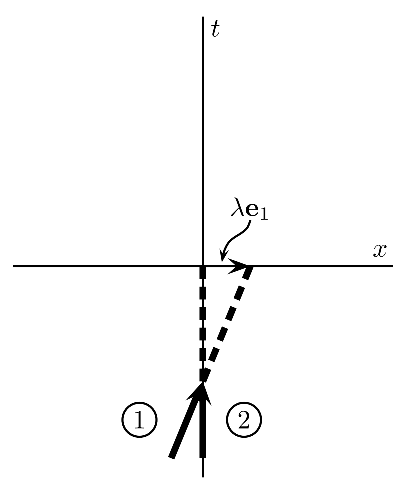

Figure 2. Case III: SAC

3 Angular Momentum Expansion

To prove our theorem for two-body scattering, we use some properties of the partial wave expansion, which we now describe.

First, we need the angular momentum expansion for two-particle wave functions. It is convenient to express $\mathbf{p}_1$ and $\mathbf{p}_2$ in terms of Gårding-Wightman variables, which for equal masses take the form

$$P = p_1 + p_2\,; \qquad Q = \frac{p_1 - p_2}{2q}\,, \quad q \equiv \tfrac{1}{2}\sqrt{s - 4m^2}\,. \tag{3.1}$$

Then $P{\cdot}Q = 0$; $Q^2 = -1$. We parametrize Q by its center of momentum values, using the boost $\Lambda(P)$ from the center of momentum to P: $Q = \Lambda(P)e$; $e = (0, \mathbf{e})$; $\mathbf{e}{\cdot}\mathbf{e} = 1$.

With these variables, the measure becomes

$$\frac{\mathrm{d}^3 p_1}{2\omega_1}\,\frac{\mathrm{d}^3 p_2}{2\omega_2} = \frac{q}{2\sqrt{s}}\,\mathrm{d}^4 P\,\mathrm{d}\Omega(\mathbf{e})\,, \tag{3.2}$$

where $\mathrm{d}\Omega(\mathbf{e})$ is the usual measure on the unit sphere, $\mathrm{d}(\cos\theta)\,\mathrm{d}\phi$ in spherical coordinates. As long as we are above the two-particle threshold, so that $q > 0$, the change of variables $\mathbf{p}_1, \mathbf{p}_2 \leftrightarrow P, \mathbf{e}$ is smooth, i.e., a function $g(P, \mathbf{e})$ is C^∞ on $\mathbb{R}^4 \times$ (unit sphere) if and only if it is smooth in $\mathbf{p}_1$, $\mathbf{p}_2$. The changes of variable $P_0 \leftrightarrow s \leftrightarrow \sqrt{s}$ are also smooth, as we are in the physical region, where $s \geq 4m^2 > 0$.

The Clebsch-Gordan coefficient for reducing the product representation $|\mathbf{p}_1, \mathbf{p}_2\rangle$ of the Poincaré group to irreducible components $|P, J, \lambda\rangle$, where $J = 0, 1, 2, \ldots$; $\lambda = -J, -J+1, \ldots, J$, and where we choose the canonical convention for the projection λ of total angular momentum in the center of momentum frame, is [23][7]

$$\langle P, J, \lambda \big| \mathbf{p}_1, \mathbf{p}_2\rangle = 2\left(\frac{s}{s-4m^2}\right)^{\frac{1}{4}} \delta^{(4)}(P - p_1 - p_2)\,\overline{Y^J_\lambda(\mathbf{e})} \tag{3.3}$$

The normalization is $\langle P, J, \lambda \big| P', J', \lambda'\rangle = \delta^{(4)}(P - P')\,\delta_{JJ'}\,\delta_{\lambda\lambda'}$, and Y^J_λ is the standard spherical harmonic.

It can be shown by more or less standard methods, using this explicit expression, that if $\langle \mathbf{p}_1, \mathbf{p}_2\,|g\rangle \in \mathcal{D}_\Delta$, where $\mathcal{D}_\Delta$ is defined in the Introduction, with the two-particle threshold excluded from Δ, then the projections onto eigenspaces of J,

$$g^J_\lambda(P) \equiv \langle P, J, \lambda | g\rangle\,, \tag{3.4}$$

are in $\mathcal{D}(\mathbb{R}^4)$, with support in Δ, and that all partial derivatives

$$D^\ell\, g^J_\lambda(P) \equiv \frac{\partial^{\ell_0}}{\partial {P_0}^{\ell_0}} \cdots \frac{\partial^{\ell_3}}{\partial {P_3}^{\ell_3}}\, g^J_\lambda(P) \tag{3.5}$$

[7] Terms with odd J are of course missing when the spinless particles are identical and parity is conserved.

vanish rapidly in J, uniformly in P. In fact, if we define seminorms (using the Schwartz notation for summation over ℓ)

$$\rho_{\Delta,L,M}(g) \equiv \sum_{\ell=0}^{L} \sum_{n=0}^{N} \sup_{J,\lambda,P} \left|(J+1)^n D^\ell g_\lambda^J(P)\right| , \tag{3.6}$$

we get a topological isomorphism between the Schwartz space $\mathcal{D}_\Delta$ and the space of sequences of C^∞ functions $g_\lambda^J(P)$, all having support in Δ, and with all derivatives decreasing rapidly in J. The expansion

$$g(\mathbf{p}_1, \mathbf{p}_2) = 2\left(\frac{s}{s-4m^2}\right)^{\frac{1}{4}} \sum_{J,\lambda} g_\lambda^J(P)\, Y_\lambda^J(\mathbf{e}) \tag{3.7}$$

converges uniformly, and rapidly in J; and the same is true of the term by term derivatives with respect to P and $\mathbf{e}$.[8]

Assuming that S is unitary and $\mathcal{P}_+$ invariant, the action of S_2 is

$$(S_2\, g)_\lambda^J(P) = \vartheta_J(s)\, g_\lambda^J(P)\,, \tag{3.8}$$

where the improper eigenvalue $\vartheta_J(s)$ of S_2 is a Lebesgue measureable function of s,[9] with $|\vartheta_J(s)| \leq 1$. To see that rapid decrease of $F(\tau) = \langle f \,|\, S \exp iM\tau\, |g\rangle$ for all $f, g \in \mathcal{D}_\Delta$ implies that $\vartheta_J(s)$ is C^∞ for s in the image of Δ, just choose f and g to be eigenstates of J, of the form

$$g^J(\mathbf{p}_1, \mathbf{p}_2) = 2\left(\frac{s}{s-4m^2}\right)^{\frac{1}{4}} \sum_{\lambda} g_\lambda^J(P)\, Y_\lambda^J(\mathbf{e})\,. \tag{3.9}$$

Then

$$\left\langle f^J \middle| S\, e^{iM\tau} \middle| g^J \right\rangle = \int \mathrm{d}^4P\; e^{i\tau\sqrt{s}}\; \vartheta_J(s) \;\sum_\lambda \overline{f_\lambda^J(P)}\, g_\lambda^J(P)\,. \tag{3.10}$$

Making the C^∞ change of variables $P \leftrightarrow (\sqrt{s}, \mathbf{P})$, we choose the sum of products of wave functions to have the form $\varphi(s)\, \psi(\mathbf{P})\, P_0/\sqrt{s}$, where φ and ψ are smooth; and we do the $\mathbf{P}$ integration, bringing the integral to the form

$$C \int \mathrm{d}(\sqrt{s})\; e^{i\tau\sqrt{s}}\; \vartheta_J(s)\; \varphi(s)\,. \tag{3.11}$$

Rapid decrease in τ implies that $\vartheta_J(s)$ is C^∞ in $\sqrt{s}$, and hence in s, from standard theorems on Fourier transforms. That holds for all smooth φ with support in the image of Δ, so $\vartheta_J(s)$ is smooth there, too.

[8]We have not found these particular results in the literature, although they are of a type which is familiar in the theory of distributions and nuclear spaces. Analogous statements about rapid convergence of the Legendre series for analytic functions are well known among physicists. The proof for nonequal masses and spins, taking threshold behavior into account, too, will be presented in another paper. Rapid convergence for Gaussian wave packets has been discussed by S. MacDowell, R. Roskies, and B. Schroer in [24].

[9]A. Martin in [16] attributes this general result to R. Stora (unpublished).

In the next section, we investigate

$$h(\mathbf{p}_1,\mathbf{p}_2) \equiv (S_2\, g)(\mathbf{p}_1,\mathbf{p}_2) = 2\left(\frac{s}{s-4m^2}\right)^{\frac{1}{4}} \sum_{J,\lambda} \vartheta_J(s)\; g^J_\lambda(P)\; Y^J_\lambda(\mathbf{e})\,, \tag{3.12}$$

and show that if $F(\tau)$ decreases rapidly, h is smooth. We can already say that h is continuous, because $\vartheta_J(s)$ is continuous and uniformly bounded in s and J, and thus does not affect the rapid, uniform convergence of the series for g. In fact, if we choose $(s,\mathbf{P},\mathbf{e})$ as variables, the same argument tells us that h is C^∞ in $(\mathbf{P},\mathbf{e})$, because the expansion still converges uniformly and rapidly, as long as ϑ_J is not differentiated. If we knew that the derivatives of ϑ_J were bounded by polynomimals in J, uniformly in s, we could argue in a similar way to show that h is C^∞, period. We do not know a direct way to get that information about ϑ_J, so we proceed by another method in Section 4.

4 Proof of Theorem 1

The proof that rapid decrease of $F(\tau)$ implies smoothness of $h = S_2\, g$ involves some rather technical steps, not all very interesting. We have tried to give only enough details to get the ideas across. The proof for the smoothness of $S_2^\dagger f$ is the same, so at least we can forget about that.

The method is based on the following lemma (Ruelle[10]): Let $x \in \mathbb{R}^s$ and $y \in \mathbb{R}^t$, and let $f(x,y)$ be Lebesgue square integrable in $\mathbb{R}^{s+t}$, i.e., $f \in L_2(\mathbb{R}^{s+t})$. Let all partial derivatives with respect to x, and separately, all derivatives with respect to y, be square integrable:

$$D^\ell_x\, f(x,y) \in L_2(\mathbb{R}^{s+t})\,, \qquad D^\ell_y\, f(x,y) \in L_2(\mathbb{R}^{s+t})\,.$$

Then f is C^∞, i.e., all mixed derivatives exist, too, and all derivatives are continuous.

To apply the lemma, we choose variables s, $\mathbf{P}$, $\mathbf{e}$, and separate them into two classes, s and $\mathbf{P}$, $\mathbf{e}$. We use the same symbol as before for h in terms of these variables, $h(s,\mathbf{P},\mathbf{e})$; and we remember that we showed from the angular momentum expansion in the last section that all derivatives of h with respect to $\mathbf{P}$, $\mathbf{e}$ are continuous. They certainly have compact support, so they are L_2.

What remains is essentially to show that $\partial^n h/\partial s^n$ is L_2. Actually, we define a new function $H \equiv h\, P_0/\sqrt{s}$, and show that it is smooth, from which we can easily recover the smoothness of h. Clearly the derivatives of H with respect to $\mathbf{P}$ and $\mathbf{e}$ are continuous with compact support, because those of h are. We need several steps to show that $\partial^n H/\partial s^n$ is L_2 for all n, and hence that H is smooth.

[10]We don't know whether this is a classical result. We brought up the question in a private conversation with D. Ruelle (1964), who provided a two or three line proof. The reader might enjoy reproducing it.

1. We begin with a weaker statement, that $\partial^n H/\partial s^n$ is defined for fixed s as a Schwartz distribution in the variables $\mathbf{P}$, $\mathbf{e}$. In fact, it follows directly from our hypothesis that it is a (weakly) continuous family of such distributions, as s varies. Let $\varphi(\mathbf{P}, \mathbf{e})$ be a smooth test function, with compact support, and define

$$\left(H_s,\ \varphi\right) = \int \mathrm{d}^3 P\ \mathrm{d}\Omega\ H(s, \mathbf{P}, \mathbf{e})\ \varphi(\mathbf{P}, \mathbf{e})\,. \tag{4.1}$$

As a function of s, the integral $\left(H_s,\ \varphi\right)$ is continuous, because the integrand is continuous, with compact suppport. Now choose a function $\psi(s) \in \mathcal{D}(\mathbb{R})$ which is unity in the support of H. From the way we defined H, including the Jacobian for the transformation from P_0 to $\sqrt{s}$, the hypothesis of the theorem says that

$$\int \mathrm{d}(\sqrt{s})\ e^{i\tau\sqrt{s}}\left(H_s,\ \varphi\right)\ \psi(s) = \int \mathrm{d}(\sqrt{s})\ e^{i\tau\sqrt{s}}\left(H_s,\ \varphi\right) \tag{4.2}$$

decreases rapidly; hence $\left(H_s,\ \varphi\right)$ is C^∞, either as a function of $\sqrt{s}$ or as a function of s, because it is the Fourier transform of a function of rapid decrease.

A standard result of distribution theory[11] tells us that $\partial^n\left(H_s,\ \varphi\right)/\partial s^n$ defines a continuous linear function on the test functions φ, which is what we claimed.

2. Because H is continuous with compact support, and C^∞ in the variables $\mathbf{P}$ and $\mathbf{e}$, $H_s(\mathbf{P}, \mathbf{e}) \equiv H(s, \mathbf{P}, \mathbf{e})$ is a test function in those variables for fixed s. Moreover, if T is any distribution in $\mathbf{P}$, $\mathbf{e}$ with compact support, then $\left(T,\ H_s\right)$ is a continuous function of s. The reader can easily verify that by representing T as a finite sum of derivatives of continuous functions, and integrating by parts.

3. Putting these two steps together, we can treat $\partial^n \overline{H}/\partial s^n$ as a weakly continuous family of distributions,[12] and H_s as a weakly continuous family of test functions. In other words, the quantities

$$\frac{\partial^n}{\partial s^n}\left(\overline{H}_s,\ H_{s'}\right)$$

are separately continuous in s and s'. These parameters vary in a finite dimensional vector spce, so by the theorem on *hypocontinuité*[13] we have not only separate but joint continuity.

4. From the symmetry $\left(\overline{H}_s,\ H_{s'}\right) = \left(H_{s'},\ \overline{H}_s\right)$, it follows that

$$\frac{\partial^n}{\partial s'^n}\left(\overline{H}_s,\ H_{s'}\right)$$

[11] L .Schwartz [14], vol. I, Théorème XIII, p. 74.

[12] The arguments in steps 1 and 2 are invariant under distribution of complex conjugates.

[13] L. Schwartz [14], vol. I, Théorème XI, p. 73.

is also jointly continuous in s and s'. In other words $\left(\overline{H}_s,\ H_{s'}\right)$ is separately C^∞ in s and s', the derivatives being jointly continuous with compact support, and hence L_2. Ruelle's lemma therefore tells us that

$$\left(\overline{H}_s,\ H_{s'}\right) \in \mathcal{D}(\mathbb{R}^2)\,. \tag{4.3}$$

5. Hence

$$\left(\frac{\partial^n \overline{H}_s}{\partial s^n},\ \frac{\partial^n H_s}{\partial s^n}\right) = \frac{\partial^n}{\partial s^n}\frac{\partial^n}{\partial s'^n}\left(\overline{H}_s,\ H s'\right)\Big|_{s=s'} \tag{4.4}$$

is in $\mathcal{D}(\mathbb{R})$, and the iterated integral

$$\int \mathrm{d}s \int \mathrm{d}^3 P\ \mathrm{d}\Omega \left|\frac{\partial^n H}{\partial s^n}(s, \mathbf{P}, \mathbf{e})\right|^2$$

is finite.

6. That completes the proof that $\partial^n H/\partial s^n$ is L_2 in all variables, and hence that H is C^∞, except for a technical point. Does the existence of the iterated integral above imply the existence of the multiple integral, so that we really have square integrability in all variables together? The rigorous answer in this case is "yes," but we refer elsewhere for the proof [17]. It requires a further look at the angular momentum expansion and its convergence almost everywhere, and an application of standard theorems on integration.

5 Conclusion

To summarize the situation, if we remember that macrocausality follows in case I from $\mathcal{P}_+$ invariance and unitarity, the basic theorem of this paper asserts that macrocausality in cases II and III is equivalent to permanence of smoothness under S, for two-particle reactions. We argued that some such regularity property is also essential for the intuitive justification of macrocausality in case III. Even that much smoothness, which is considerably less than that suggested by perturbation theory, is sufficiently strong to have resisted proof in field theory; and it has been suggested [16] that field theory may not have enough dynamical content, without further postulates, to permit the proof of one of its consequences, smoothness of the two-body partial waves in the elastic region. One could thus legitimately question whether rapid decrease is "common sense," at least in case III, although surely few would disagree that smoothness (or even analyticity) between thresholds is a reasonable property for the two-particle scattering amplitude to have.

Acknowledgments

I have benefited a great deal from a collaboration with J. Bros, which led to the problems discussed here. It is a pleasure to thank Dr. J.-J. Loeffel and Dr. J. Nuyts for several helpful conversations. I am also indebted to the late Dr. J. Robert Oppenheimer and to Dr. Carl Kaysen for the hospitality of the Institute for Advanced Study.

References

[1] R. E. Norton, *Phys. Rev.* **135**, B1381 (1964).

[2] R. E. Norton and S. Coleman, *Nuovo Cimento* **38**, 438 (1965).

[3] F. Pham, "Singularités de Processus de Diffusion Multiple," I and II, CERN preprints.

[4] C. Chandler and H. P. Stapp, "Macroscopic Causality Conditions and Properties of Scattering Amplitudes," Lawrence Radiation Laboratory preprint, UCRL–17734.

[5] E. H. Wichmann and J. H. Crichton, *Phys. Rev.* **132**, 2788 (1963).

[6] K. Hepp, *Helv. Phys. Acta* **37**, 659 (1964).

[7] H. P. Stapp, *Phys. Rev.* **139**, B257 (1965).

[8] K. Hepp, *J. Math. Phys.* **6**, 1762 (1965).

[9] A. Peres, *Annals of Physics* **37**, 179 (1966).

[10] R. P. Feynman, *Phys. Rev.* **76**, 749, 769 (1949); E. C. G. Stueckelberg and D. Rivier, *Helv. Phys. Acta* **23**, 215 (1950); M. Fierz, *Helv. Phys. Acta* **23**, 731 (1950); W. Zimmerman, *Nuovo Cimento* **13**, 503 (1959) and **16**, 690 (1960); D. Branson, *Phys. Rev.* **135**, B1255 (1964); R. J. Eden and P. V. Landschoff, *Annals of Physics* **31**, 370 (1965); D. Iagolnitzer, *J. Math. Phys.* **6**, 1576 (1965); G. Wanders, *Helv. Phys. Acta* **38**, 142 (1965).

[11] M. L. Goldberger and K. M. Watson, *Phys. Rev.* **127**, 2284 (1962); M. Froissart, M. L. Goldberger, and K. M. Watson, *Phys. Rev.* **131**, 2820 (1963).

[12] R. Haag, *Phys. Rev.* **112**, 669 (1958).

[13] D. Ruelle, *Helv. Phys. Acta* **35**, 147 (1962).

[14] L. Schwartz, *Théorie des Distributions*, vols. I, II, Hermann, Paris, 1957, 1959.

[15] R. Jost and K. Hepp, *Helv. Phys. Acta* **35**, 34 (1962).

[16] A. Martin, "Inability of Field Theory to Exploit the Full Unitarity Condition," CERN preprint, 1966.

[17] D. Williams, to be submitted for publication.

[18] K. Hepp, *Commun. Math. Phys.* **1**, 95 (1965).

[19] T. D. Newton and E. P. Wigner, *Rev. Mod. Phys.* **21**, 400 (1949).

[20] E. Nelson, *Ann. Math.* **70**, 572 (1959).

[21] L. Gårding, *Proc. Nat. Acad. Sci. USA* **33**, 331 (1947).

[22] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Ungar, New York, 1955, p. 301.

[23] H. Joos, *Fortschr. Physik* **10**, 65 (1962).

[24] S. MacDowell, R. Roskies, and B. Schroer, "Cluster Properties and Bounds on the Scattering Amplitude," Yale University preprint, 1967.