Agenda:

1. Hypothesis testing under normality
2. Confidence intervals
3. t-test

Q: What is the distribution of the OLS estimator $\hat{\beta}$?
(under the linear model w/ Gaussian/normal error terms)

C1 (linearity): there is a vector of regression coefficients

$\beta^* \in \mathbb{R}^p$ s.t $y = X\beta^* + \epsilon$

C2 (normality of error terms): $\epsilon | X \sim N(0, \sigma^2 I_n)$

Consequence of C1 + C2:

$$E[y|X] = E[X\beta^* + \epsilon | X] \quad \text{(linearity)}$$
$$= X\beta_* + \cancel{E[\epsilon|X]}^{0}$$
$$= X\beta_* \quad \text{(normality of error terms)}$$

Claim: $\hat{\beta} - \beta^* | X \sim N(0, \sigma^2 (X^T X)^{-1})$

Recall if $z \sim N(\mu, \Sigma)$ then $Az \sim N(A\mu, A\Sigma A^T)$

$$\hat{\beta} = (X^T X)^{-1} X^T y$$
$$= (X^T X)^{-1} X^T (X\beta^* + \epsilon)$$
$$= \cancel{(X^T X)^{-1}} \cancel{X^T X} \beta^* + (X^T X)^{-1} X^T \epsilon$$

$= \beta^* + (X^TX)^{-1}X^T\epsilon$

$\hat{\beta} - \beta^* = (X^TX)^{-1}X^T\epsilon$

C2 : $\epsilon \mid X \sim N(0, \boxed{\sigma^2 I_n})$

$\underbrace{(X^TX)^{-1}X^T}_{A}\epsilon \mid X \sim N(0, \underbrace{\underbrace{(X^TX)^{-1}X^T}_{A}(\sigma^2 I_n)\underbrace{X(X^TX)^{-1}}_{A^T}})$

$= \sigma^2 \cancel{(X^TX)^{-1}} \cancel{X^T} X (X^TX)^{-1}$

Inference under normality

- Hypothesis testing
- confidence intervals

Testing $H_0: \beta_j^* = 0$

1. Cook up a test statistic & derive its distribution under H_0 (and linearity + normality of error terms)

2. Ascertain how "surprising" the observed value of the test statistic is.

For testing $H_0: \beta_j^* = 0$

pick $\hat{\beta}_j$ as my test statistic

$\hat{\beta} - \beta^* \mid X \sim N(0, \sigma^2 (X^TX)^{-1})$

$\hat{\beta}_j - \beta_j^* \mid X \sim N(0, \sigma^2 \underbrace{[(X^TX)^{-1}]_{j,j}}_{j\text{-th diagonal entry}})$

Aside :

If $z \sim N(\mu, \Sigma)$

then $z_j \sim N(\mu_j, [\Sigma]_{j,j})$

of $(X^TX)^{-1}$

Under $H_0: \beta_j^* = 0$

$$\hat\beta_j \mid X \sim N\Big(0, \underbrace{\sigma^2\big[(X^TX)^{-1}\big]_{j,j}}_{SE(\hat\beta_j)^2}\Big)$$

$$z_j = e_j^T z, \quad e_j : \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow j\text{-th entry}$$

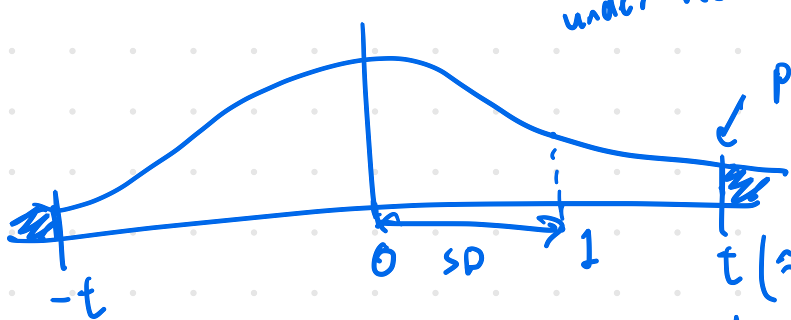
$$e_j^T z \sim N(\underbrace{e_j^T \mu}_{\mu_j}, e_j^T \Sigma e_j)$$

$$\frac{\hat\beta_j}{SE[\hat\beta_j]} \Big| X \sim N(0,1)$$

Intuition: $\left\{\begin{array}{l} \text{reject } H_0: \beta_j^0 = 0 \\ \text{claim } H_0 \text{ is false} \end{array}\right\}$ if $\left|\frac{\hat\beta_j}{SE[\hat\beta_j]}\right| \geq t$, when t is a CAREFULLY CHOSEN threshold

Want to pick t s.t. $P_{\boxed{0}}\left\{\left|\frac{\hat\beta_j}{SE[\hat\beta_j]}\right| \geq t\right\} \leq \boxed{\alpha}$ ← usually 0.05

for some prescribed $\alpha \geq 0$ ↑ probability under H_0



pdf of $\frac{\hat\beta_j}{SE[\hat\beta_j]}$

t (≈ 1.96 for $\alpha = 0.05$)

Pick t st. shaded area is at most α (usually 0.05).

Confidence intervals:

rejection event : $\left\{\left|\frac{\hat\beta_j}{SE[\hat\beta_j]}\right| > 2\right\} \iff \left\{0 \notin \boxed{\left[\hat\beta_j - 2SE[\hat\beta_j], \hat\beta_j + 2SE[\hat\beta_j]\right]}\right\}$

(for $\alpha = 0.05$)

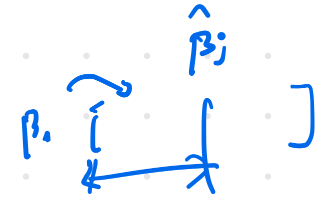
confidence interval

If $\frac{\hat\beta}{SE[\hat\beta_j]} > 2$, then $\hat\beta_j > 2SE[\hat\beta_j]$

$\hat\beta_j - 2SE[\hat\beta_j] > 0$

If $\frac{\hat\beta}{SE[\hat\beta_j]} < -2$, then $\hat\beta < -2SE[\hat\beta_j]$

$\hat\beta + 2SE[\hat\beta_j] < 0$



$P\{\beta^0 \in [\hat\beta_j - 2SE[\hat\beta_j], \hat\beta_j + 2SE[\hat\beta_j]]\}$

$$= P\left\{ \hat{\beta}_j - \beta_j^* \in \left[-2SE[\hat{\beta}_j] \,,\, 2SE[\hat{\beta}_j] \right] \right\}$$

$$= P\left\{ \underbrace{\frac{\hat{\beta}_j - \beta_j^*}{SE[\hat{\beta}_j]}}_{N(0,1)} \in [-2, 2] \right\}$$

$$\approx 0.95$$